

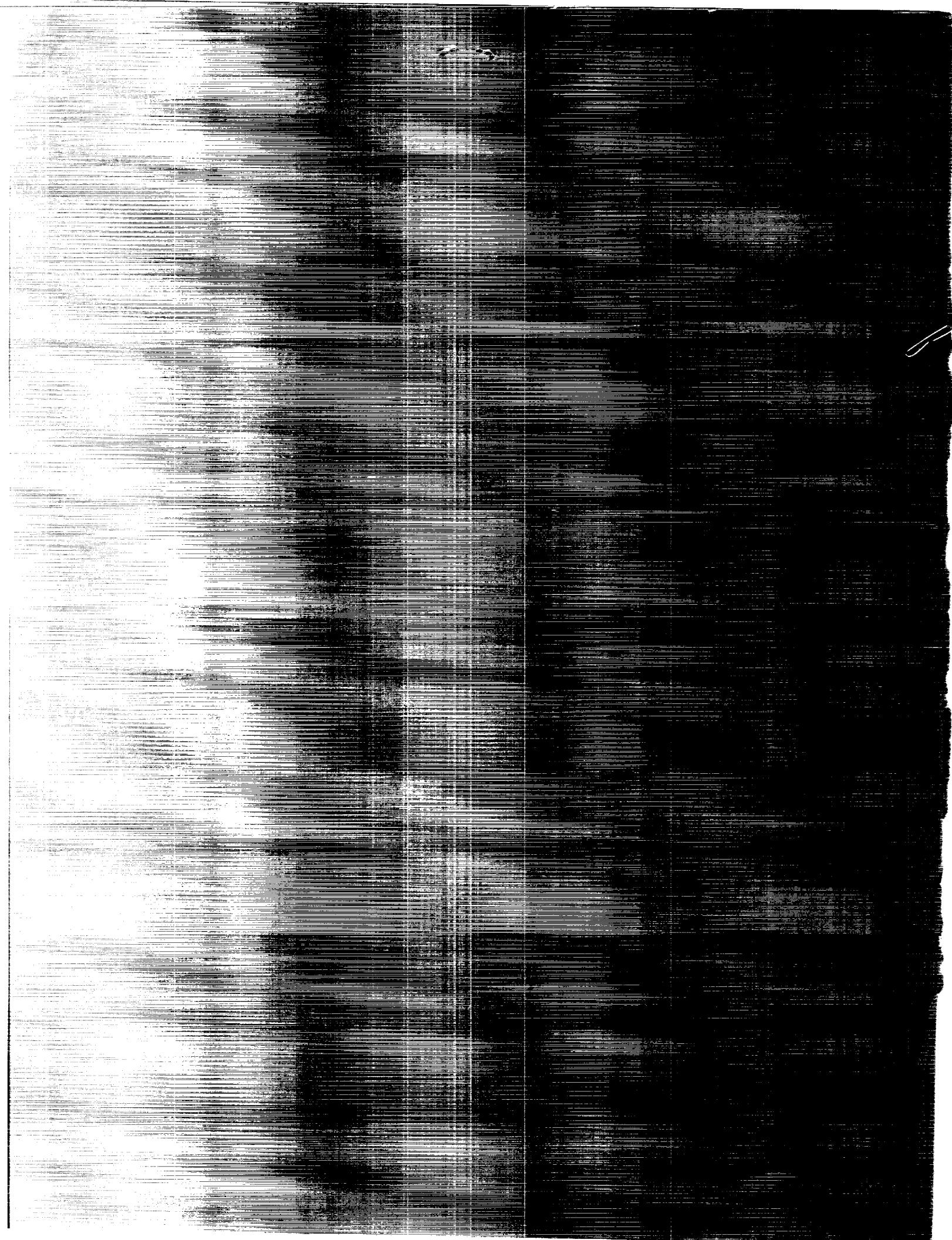
(NASA-TM-4060) DERIVATION OF REVISED
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**Derivation of Revised Formulae for
Eddy Viscous Forces Used in the
Ocean General Circulation Model**

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In ocean general circulation models, such as Bryan-Cox¹ and other derived models, the dissipation term is usually taken to have two axes of symmetry, with eddy coefficients A_L and A_H in the symmetry plane and symmetry axis direction respectively. A_L is much greater than A_H , and both are much greater than the molecular value. If we further impose the condition of complete isotropy, the viscous stress term must return to the usual Laplacian of the velocity multiplied by a scalar, in whatever coordinates the equations are written and in particular in the earth surface coordinates in which the codes are written. However, the viscous stress terms used by nearly all the models, which were taken from Kamenkovich, do not do this. The reason for this flaw is the Kamenkovich's linearization of the gauge matrix occurred too early, i.e., the substitution of the earth radius a for the radius $r = a + z$ was made too early, so that some of the z-derivatives that should appear failed to appear.

In this paper, the correct form of the viscous terms are presented. Indeed, the practical consequences of the error is probably not too serious, since the omitted z-derivatives in Bryan-Cox¹ are multiplied by A_H , and are therefore smaller compared to the terms multiplied by A_L . Nevertheless, we present this paper in the interest of consistency and possible future use. In the following, we first show our results in rectangular coordinates; then a revised form for the turbulent viscous term in earth-surface coordinates will be derived. The detailed formulae are given in the appendixes.

1. Molecular and Eddy Viscosity in Rectangular Coordinates

The eddy viscous force for incompressible fluids is the divergence of the Reynolds' stress (Appendix I)

$$F_i = \frac{\partial}{\partial x_j} R_{ij} \quad (1.1)$$

an assumption widely accepted today. In (1.1), subscripts i and j stand for the three Cartesian components.

The two horizontal components of (1.1) can be written as²

$$\frac{1}{\rho_0} F_x = A_L \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \frac{\partial}{\partial z} A_H \frac{\partial u}{\partial z} + \frac{\partial}{\partial z} A_H \frac{\partial w}{\partial x} - \frac{\partial}{\partial x} A \frac{\partial w}{\partial z}$$

and

(1.2)

$$\frac{1}{\rho_0} F_y = A_L \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) + \frac{\partial}{\partial z} A_H \frac{\partial v}{\partial z} + \frac{\partial}{\partial z} A_H \frac{\partial w}{\partial y} - \frac{\partial}{\partial y} A \frac{\partial w}{\partial z}$$

where u , v , and w are the x , y and z components of velocity; A , A_H and A_L are the turbulence viscosity coefficients and they can be functions of x , y , z .

For 3-dimensional isotropic fluids, i.e. $A = A_H = A_L = \text{constant} = \kappa/\rho_0$, the Reynolds' stress becomes:

$$R_{ij} = \kappa \left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j} \right) \quad (1.3)$$

and the eddy viscous term is then

$$F_i = \kappa \frac{\partial}{\partial x_j} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (1.4)$$

For incompressible fluids,

$$\frac{\partial u_i}{\partial x_i} = 0 \quad (1.5)$$

(1.4) becomes

$$F_i = \kappa \frac{\partial}{\partial x_j} \frac{\partial u_i}{\partial x_j} \quad (1.6)$$

which of course can also be obtained by directly inserting a constant turbulence coefficient into (1.2).

Obviously, under the complete isotropic conditions, this eddy viscosity term has the same form as molecular viscosity term – Laplacian of the velocity, the only difference being that the eddy viscosity coefficient is much larger than the molecular value.

2. Eddy and Molecular Viscosity Terms in Earth Surface Coordinates

In rectangular coordinates, we have seen that when the transverse symmetric condition is replaced by the complete isotropic condition, the formulae for the eddy viscosity term returns to its spherical symmetric form. This is easily seen in rectangular coordinates because the gauge matrix is constant. Choosing λ , longitude, ϕ , latitude, and z , distance along the earth radius from the earth surface, we get the earth surface coordinates. These are curvilinear coordinates in which the Reynolds' stress (Appendix II) and the corresponding viscosity terms have more complicated forms; this fact, however, does not change the previous conclusion, since any physical phenomenon should not depend on coordinates chosen.

In earth surface coordinates, the components of the gauge matrix are

$$\begin{aligned} h_\lambda &= (z + a) \cos \phi \\ h_\phi &= (z + a) \\ h_z &= 1 \\ h &= (z + a)^2 \cos \phi \end{aligned} \quad (2.1)$$

where a is the radius of the earth.

The force F_α , divergence of the Reynolds' tensor $R_{\alpha\beta}$, is now

$$F_\alpha = \frac{1}{h_\alpha} \sum_\beta \left(\frac{1}{h} \frac{\partial}{\partial q_\beta} \left[\frac{h h_\alpha}{h_\beta} R_{\alpha\beta} \right] - \frac{R_{\beta\beta}}{h_\beta} \frac{\partial h_\beta}{\partial q_\alpha} \right) \quad (2.2)$$

whose two horizontal components are:

$$\begin{aligned} F_\lambda &= \frac{1}{h h_\lambda} \left(\frac{\partial}{\partial \lambda} \left[h R_{\lambda\lambda} \right] + \frac{\partial}{\partial \phi} \left[\frac{h_\lambda h}{h_\phi} R_{\phi\lambda} \right] + \frac{\partial}{\partial z} \left[\frac{h_\lambda h}{h_z} R_{z\lambda} \right] \right) \\ F_\phi &= \frac{1}{h h_\phi} \left(\frac{\partial}{\partial \lambda} \left[\frac{h_\phi h}{h_\lambda} R_{\phi\lambda} \right] + \frac{\partial}{\partial \phi} \left[h R_{\phi\phi} \right] + \frac{\partial}{\partial z} \left[\frac{h_\phi h}{h_z} R_{z\phi} \right] \right) - \frac{R_{\lambda\lambda}}{h_\lambda h_\phi} \frac{\partial h_\lambda}{\partial \phi} \end{aligned} \quad (2.3)$$

Inserting the gauge matrix (2.1), the Reynolds' stress, the condition of incompressible fluids

$$\operatorname{div} \vec{v} = \frac{1}{h} \left(\frac{\partial}{\partial \lambda} [h_\phi h_z u_\lambda] + \frac{\partial}{\partial \phi} [h_\lambda h_z u_\phi] + \frac{\partial}{\partial z} [h_\lambda h_\phi u_z] \right) = 0 \quad (2.4)$$

and its related formulae shown in Appendix III into eq. (2.3), we obtain a set of viscosity forces in the earth surface coordinate:

$$\begin{aligned} \frac{1}{\rho_0} F_\lambda &= A_L \left(\Delta_2 u_\lambda + \frac{1 - \tan^2 \phi}{(z+a)^2} u_\lambda - \frac{2 \tan \phi}{(z+a)^2 \cos \phi} \frac{\partial u_\phi}{\partial \lambda} \right) \\ &+ \frac{1}{h} \frac{\partial}{\partial z} \left(\frac{h}{h_z^2} A_H \frac{\partial u_\lambda}{\partial z} \right) + \frac{2A_H}{(z+a)^2 \cos \phi} \frac{\partial u_z}{\partial \lambda} - \frac{2A_H}{(z+a)^2} u_\lambda - \frac{u_\lambda}{z+a} \frac{\partial A_H}{\partial z} \\ &+ \frac{A_H}{(z+a) \cos \phi} \frac{\partial^2 u_z}{\partial z \partial \lambda} + \frac{1}{(z+a) \cos \phi} \frac{\partial u_z}{\partial \lambda} \frac{\partial A_H}{\partial z} \\ &- \frac{A}{(z+a)^2 \cos \phi} \frac{\partial^2 u_z}{\partial z \partial \lambda} - \frac{1}{(z+a) \cos \phi} \frac{\partial A}{\partial \lambda} \frac{\partial u_z}{\partial z} \end{aligned} \quad (2.5)$$

and

$$\begin{aligned} \frac{1}{\rho_0} F_\phi &= A_L \left(\Delta_2 u_\phi + \frac{1 - \tan^2 \phi}{(z+a)^2} u_\phi + \frac{2 \tan \phi}{(z+a)^2 \cos \phi} \frac{\partial u_\lambda}{\partial \lambda} \right) \\ &+ \frac{1}{h} \frac{\partial}{\partial z} \left(\frac{h}{h_z^2} A_H \frac{\partial u_\phi}{\partial z} \right) + \frac{2A_H}{(z+a)^2} \frac{\partial u_z}{\partial \phi} - \frac{2A_H}{(z+a)^2} u_\phi - \frac{u_\phi}{z+a} \frac{\partial A_H}{\partial z} \\ &+ \frac{A_H}{z+a} \frac{\partial^2 u_z}{\partial z \partial \phi} + \frac{1}{z+a} \frac{\partial u_z}{\partial \phi} \frac{\partial A_H}{\partial z} \\ &- \frac{A}{z+a} \frac{\partial^2 u_z}{\partial z \partial \phi} - \frac{1}{z+a} \frac{\partial u_z}{\partial z} \frac{\partial A}{\partial \phi} \end{aligned} \quad (2.6)$$

where Δ_2 stands for the 2-dimensional (λ and ϕ) Laplacian operator, and A , A_H and A_L are the three eddy viscosity coefficients.

If we let $A = A_H = A_L = \text{constant}$, we will obtain the viscous forces under the isotropic conditions:

$$\frac{1}{\rho_0} F_\lambda = A \left(\Delta u_\lambda - \frac{1 + \tan^2 \phi}{(z+a)^2} u_\lambda - \frac{2 \tan \phi}{(z+a)^2 \cos \phi} \frac{\partial u_\phi}{\partial \lambda} + \frac{2}{(z+a)^2 \cos \phi} \frac{\partial u_z}{\partial \lambda} \right)$$

and

$$\frac{1}{\rho_0} F_\phi = A \left(\Delta u_\phi - \frac{1 + \tan^2 \phi}{(z+a)^2} u_\phi + \frac{2 \tan \phi}{(z+a)^2 \cos \phi} \frac{\partial u_\lambda}{\partial \lambda} + \frac{2}{(z+a)^2} \frac{\partial u_z}{\partial \phi} \right) \quad (2.7)$$

These results are the same as that obtained from the direct transformation of the Laplacian from ordinary spherical coordinates to earth-surface coordinates. These terms, in fact, are identical to the molecular viscosity terms except the eddy viscous coefficients.

Since the thin shell approximation, $z \ll a$, is used, z is usually taken to be zero in the denominators of eq. (2.7); in addition, u_z is much smaller than u_ϕ and u_λ , so it and its derivatives are omitted in these formulae as well. Then the correct approximation for the turbulent viscosity should be :

$$\begin{aligned} \frac{1}{\rho_0} F_\lambda &= A_L \left(\Delta_2 u_\lambda + \frac{1 - \tan^2 \phi}{a^2} u_\lambda - \frac{2 \tan \phi}{a^2 \cos \phi} \frac{\partial u_\phi}{\partial \lambda} \right) \\ &+ \frac{1}{h} \frac{\partial}{\partial z} \left(\frac{h}{h_z^2} A_H \frac{\partial u_\lambda}{\partial z} \right) - \frac{2A_H}{a^2} u_\lambda \end{aligned} \quad (2.8)$$

$$\frac{1}{\rho_0} F_\phi = A_L \left(\Delta_2 u_\phi + \frac{1 - \tan^2 \phi}{a^2} u_\phi + \frac{2 \tan \phi}{a^2 \cos \phi} \frac{\partial u_\lambda}{\partial \lambda} \right) \\ + \frac{1}{h} \frac{\partial}{\partial z} \left(\frac{h}{h_z^2} A_H \frac{\partial u_\phi}{\partial z} \right) - \frac{2A_H}{a^2} u_\phi \quad (2.9)$$

with the constant viscosity coefficients.

If we now impose the isotropic condition, the viscous forces (2.9) become

$$\frac{1}{\rho_0} F_\lambda = A \left(\Delta u_\lambda - \frac{1 + \tan^2 \phi}{a^2} u_\lambda - \frac{2 \tan \phi}{a^2 \cos \phi} \frac{\partial u_\phi}{\partial \lambda} \right) \\ \frac{1}{\rho_0} F_\phi = A \left(\Delta u_\phi - \frac{1 + \tan^2 \phi}{a^2} u_\phi + \frac{2 \tan \phi}{a^2 \cos \phi} \frac{\partial u_\lambda}{\partial \lambda} \right) \quad (2.10)$$

But when we impose these two conditions (isotropy and thin shell) into the equations in Ref. 2, we obtain instead

$$\frac{1}{\rho_0} F_\lambda = A \left(\Delta u_\lambda + \frac{1 - \tan^2 \phi}{a^2} u_\lambda - \frac{2 \tan \phi}{a^2 \cos \phi} \frac{\partial u_\phi}{\partial \lambda} \right)$$

and

$$\frac{1}{\rho_0} F_\phi = A \left(\Delta u_\phi + \frac{1 - \tan^2 \phi}{a^2} u_\phi + \frac{2 \tan \phi}{a^2 \cos \phi} \frac{\partial u_\lambda}{\partial \lambda} \right) \quad (2.11)$$

which are different from eq. (2.10); in other words, the equations in Ref. 2, under the isotropic condition, do not return to a spherical symmetrical form. This is why the equations in Ref. 2 are not a proper approximation for the eddy turbulent viscosity terms, although the last terms

$$- \frac{2A_H}{a^2} u_\lambda \quad \text{in (2.8)}$$

and

$$- \frac{2A_H}{a^2} u_\phi \quad \text{in (2.9)}$$

are very small compared with other terms in the equation (due to the very large radius of earth and small A_H), they should nevertheless be retained in a complete set of eddy viscous forces.

The cause of this incompleteness is that the gauge matrix was linearized according to thin shell approximation too early, so the terms involving z-derivatives fail to appear when they should appear.

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Appendix

I. Reynolds' stress R_{ij} in rectangular coordinate^{2,3}

$$R_{xx} = -\frac{2}{3}\rho_0 b + \rho_0 \left[A_L \Phi_{xx} + \frac{1}{2} [A_L - A] \Phi_{zz} \right]$$

$$R_{yy} = -\frac{2}{3}\rho_0 b + \rho_0 \left[A_L \Phi_{yy} + \frac{1}{2} [A_L - A] \Phi_{zz} \right]$$

$$R_{zz} = -\frac{2}{3}\rho_0 b + \rho_0 A \Phi_{zz}$$

$$R_{xy} = R_{yx} = \rho_0 A_L \Phi_{xy}$$

$$R_{xz} = R_{zx} = \rho_0 A_H \Phi_{xz}$$

$$R_{yz} = R_{zy} = \rho_0 A_H \Phi_{yz} \quad (1)$$

where

$$\Phi_{ij} = \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \quad (2)$$

II. Reynolds' stress in earth surface coordinate:

$$R_{\lambda\lambda} = -\frac{2}{3}\rho_0 b + \rho_0 \left[A_L \Phi_{\lambda\lambda} + \frac{1}{2} [A_L - A] \Phi_{zz} \right]$$

$$R_{\phi\phi} = -\frac{2}{3}\rho_0 b + \rho_0 \left[A_L \Phi_{\phi\phi} + \frac{1}{2} [A_L - A] \Phi_{zz} \right]$$

$$R_{zz} = -\frac{2}{3}\rho_0 b + \rho_0 A \Phi_{zz}$$

$$\begin{aligned}
R_{\lambda\phi} &= \rho_0 A_L \Phi_{\lambda\phi} \\
R_{\lambda z} &= \rho_0 A_H \Phi_{\lambda z} \\
R_{\phi z} &= \rho_0 A_H \Phi_{\phi z}
\end{aligned} \tag{3}$$

where $\Phi_{\alpha\beta}$ is assumed as:

$$\Phi_{\alpha\beta} = \frac{h_\alpha}{h_\beta} \frac{\partial}{\partial q_\beta} \left(\frac{u_\alpha}{h_\alpha} \right) + \frac{h_\beta}{h_\alpha} \frac{\partial}{\partial q_\alpha} \left(\frac{u_\beta}{h_\beta} \right) + 2\delta_{\alpha\beta} \sum_\gamma \frac{u_\gamma}{h_\gamma} \frac{1}{h_\alpha} \frac{\partial h_\alpha}{\partial q_\gamma} \tag{4}$$

where q stands for the coordinates λ, ϕ, z and the summation index γ rolls over λ, ϕ, z components. We have the following expressions for its components:

$$\begin{aligned}
\Phi_{\lambda\lambda} &= 2 \frac{\partial}{\partial \lambda} \left(\frac{u_\lambda}{h_\lambda} \right) + 2 \left(\frac{u_\phi}{h_\phi} \frac{1}{h_\lambda} \frac{\partial h_\lambda}{\partial \phi} + \frac{u_z}{h_z} \frac{1}{h_\lambda} \frac{\partial h_\lambda}{\partial z} \right) \\
\Phi_{\phi\phi} &= 2 \frac{\partial}{\partial \phi} \left(\frac{u_\phi}{h_\phi} \right) + 2 \left(\frac{u_z}{h_z} \frac{1}{h_\phi} \frac{\partial h_\phi}{\partial z} \right) \\
\Phi_{zz} &= 2 \frac{\partial}{\partial z} \left(\frac{u_z}{h_z} \right) \\
\Phi_{\lambda\phi} &= \frac{h_\lambda}{h_\phi} \frac{\partial}{\partial \phi} \left(\frac{u_\lambda}{h_\lambda} \right) + \frac{h_\phi}{h_\lambda} \frac{\partial}{\partial \lambda} \left(\frac{u_\phi}{h_\phi} \right) \\
\Phi_{\lambda z} &= \frac{h_\lambda}{h_z} \frac{\partial}{\partial z} \left(\frac{u_\lambda}{h_\lambda} \right) + \frac{h_z}{h_\lambda} \frac{\partial}{\partial \lambda} \left(\frac{u_z}{h_z} \right) \\
\Phi_{\phi z} &= \frac{h_\phi}{h_z} \frac{\partial}{\partial z} \left(\frac{u_\phi}{h_\phi} \right) + \frac{h_z}{h_\phi} \frac{\partial}{\partial \phi} \left(\frac{u_z}{h_z} \right)
\end{aligned} \tag{5}$$

III. Two formulae related to the incompressible fluids condition (2.4) are very useful and are derived below:

$$\begin{aligned}
 \frac{1}{h_\lambda} \frac{\partial}{\partial \lambda} \operatorname{div} \vec{v} &= \frac{1}{h_\lambda^2} \frac{\partial^2 u_\lambda}{\partial \lambda^2} + \frac{h_z}{h_\lambda h} \frac{\partial h_\lambda}{\partial \phi} \frac{\partial u_\phi}{\partial \lambda} + \frac{h_z}{h} \frac{\partial^2 u_\phi}{\partial \lambda \partial \phi} \\
 &+ \frac{h_\phi}{h} \frac{\partial^2 u_z}{\partial z \partial \lambda} + \frac{h_\phi}{h_\lambda h} \frac{\partial h_\lambda}{\partial z} \frac{\partial u_z}{\partial \lambda} + \frac{1}{h} \frac{\partial h_\phi}{\partial z} \frac{\partial u_z}{\partial \lambda} \\
 &= 0
 \end{aligned} \tag{6}$$

$$\begin{aligned}
 \frac{1}{h_\phi} \frac{\partial}{\partial \phi} \operatorname{div} \vec{v} &= \frac{h_z}{h} \frac{\partial^2 u_\lambda}{\partial z \partial \phi} - \frac{1}{h_\lambda^2 h_\phi} \frac{\partial h_\lambda}{\partial \phi} \frac{\partial u_\lambda}{\partial \lambda} + \frac{1}{h_\phi^2} \frac{\partial^2 u_\phi}{\partial \phi^2} + \frac{1}{h_\lambda h_\phi^2} \frac{\partial u_\phi}{\partial \phi} \frac{\partial h_\lambda}{\partial \phi} \\
 &+ \frac{u_\phi}{h_\lambda h_\phi^2} \frac{\partial^2 h_\lambda}{\partial \phi^2} - \frac{u_\phi}{h_\lambda^2 h_\phi^2} \left(\frac{\partial h_\lambda}{\partial \phi} \right)^2 + \frac{1}{h_\lambda h_\phi} \frac{\partial h_\lambda}{\partial z} \frac{\partial u_z}{\partial \phi} \\
 &+ \frac{u_z}{h_\lambda h_\phi} \frac{\partial^2 h_\lambda}{\partial \phi \partial z} - \frac{u_z}{h_\lambda^2 h_\phi} \frac{\partial h_\lambda}{\partial \phi} \frac{\partial h_\lambda}{\partial z} + \frac{1}{h_\phi h_z} \frac{\partial^2 u_z}{\partial \phi \partial z} + \frac{1}{h_\phi^2 h_z} \frac{\partial h_\phi}{\partial z} \frac{\partial u_z}{\partial \phi} \\
 &= 0
 \end{aligned} \tag{7}$$

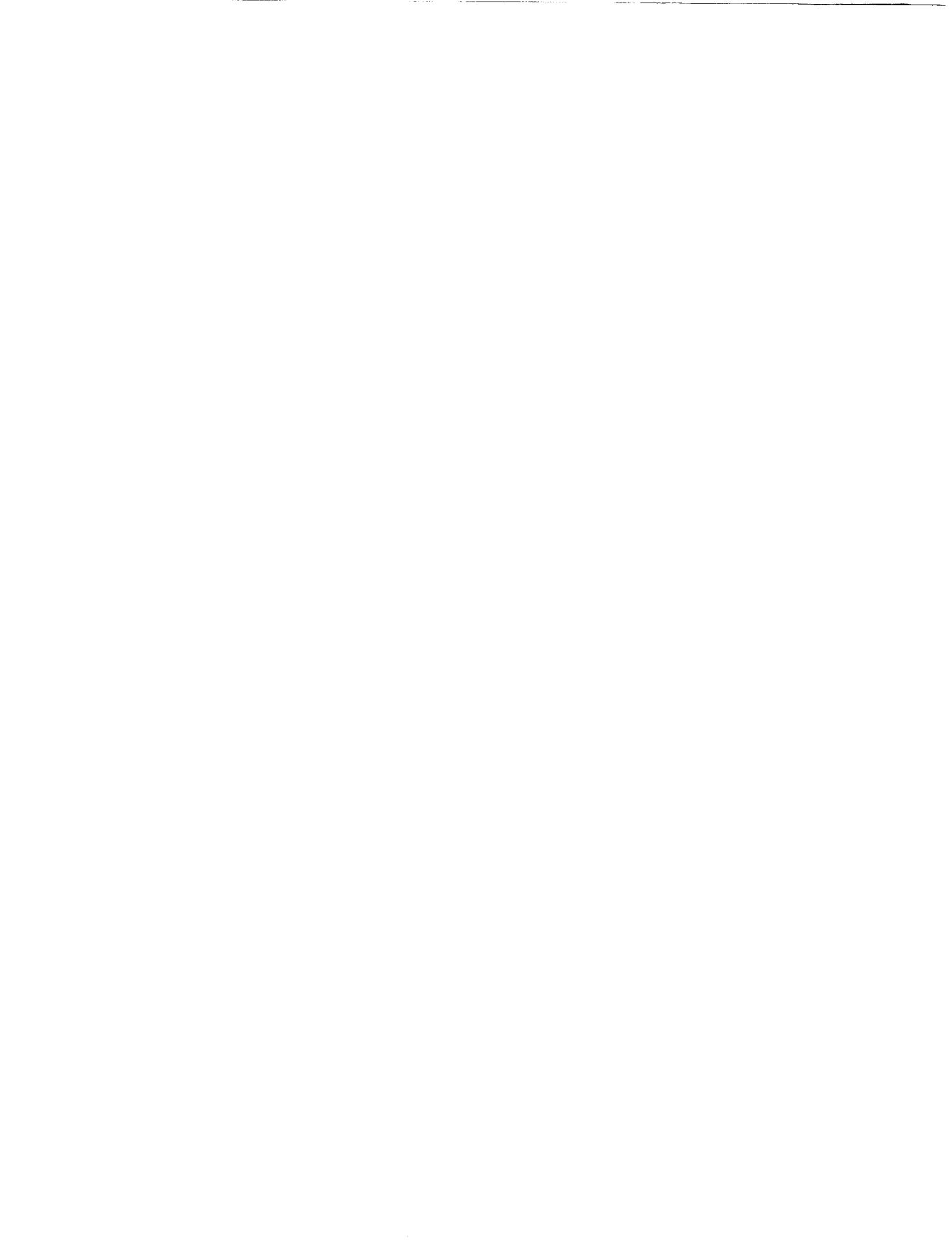
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16. Abstract <p>This paper presents a re-derivation of the eddy viscous dissipation tensor commonly used in present oceanographic general circulation models. When isotropy is imposed, the currently-used form of the tensor fails to return to the laplacian operator. In this paper, the source of this error is identified in a consistent derivation of the tensor in both rectangular and earth spherical coordinates, and the correct form of the eddy viscous tensor is presented.</p>		13. Type of Report and Period Covered	
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